Approximate Eigenvalue Distribution of a Cylindrically Isotropic Noise Sample Covariance Matrix

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Introduction

This work models the eigenvalue behavior of a sample covariance matrix (SCM) for cylindrically isotropic noise using Random Matrix Theory.

Objective:
Devise a method to approximate the asymptotic eigenvalue density function (EDF) of cylindrically isotropic noise.

Background

Assumption: An N element horizontal uniform linear array in the shallow water acoustic channel with cylindrically isotropic noise field [1].

- Ensemble noise covariance matrix $[\mathbf{\Sigma}]_{pq} = J_0(\alpha|p-q|)$ where $\alpha = 2\pi\zeta$ and ζ is the ratio of the sensor element spacing to the wavelength.
- $\mathbf{eig}(\mathbf{\Sigma}) = \gamma_1 > \gamma_2 > \ldots > \gamma_N$
- Noise realization matrix: $\mathbf{X} = \mathbf{\Sigma}^{1/2}\mathcal{G}$ where $\mathcal{G}$ is a $N \times L$ matrix of IID $\mathcal{CN}(0,1)$

Noise SCM:

$$\mathbf{S_X} = (1/L)\mathbf{\Sigma}^{1/2}(\mathcal{G}\mathcal{G}^H)\mathbf{\Sigma}^{1/2} = \mathbf{\Sigma}^{1/2}\mathbf{W}(c)\mathbf{\Sigma}^{1/2}$$

where $\mathbf{W}(c)$ is a Wishart matrix, $c = N/L$

- Insight: $\mathbf{eig}(\mathbf{\Sigma}^{1/2}\mathbf{W}(c)\mathbf{\Sigma}^{1/2}) = \mathbf{eig}(\mathbf{\Sigma}\mathbf{W}(c))$
- Allows *Multiply Wishart* operation from Polynomial Method (PM) to compute the SCM EDF [2].

$\mathbf{\Sigma}$ to be an *algebraic matrix*. Is this satisfied?

Algebraic model for Σ: Naive Approach

- Construct an atomic EDF $f^{\Sigma}(x)$ with all N eigenvalues of $\mathbf{\Sigma}$
- *Multiply Wishart* operation to obtain the polynomial $L_{mz}^{\mathbf{S_X}}$
- Extract the EDF $f^{\mathbf{S_X}}(x)$ from the roots of $L_{mz}^{\mathbf{S_X}}$

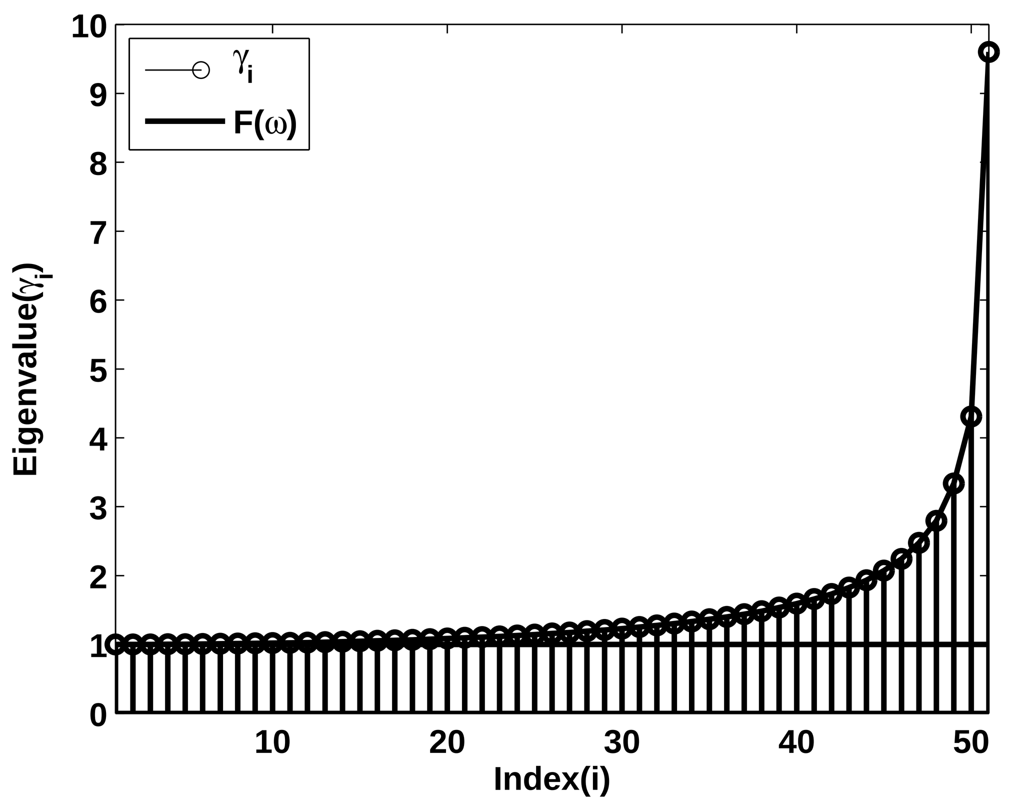
Drawback:

- Degree of polynomial $L_{mz}^{\mathbf{S_X}}$ grows as $\mathcal{O}(N)$
- Computing roots of polynomial is expensive $\mathcal{O}(N^3)$

⇒ Lower order approximation?

Ensemble covariance matrix

- $\mathbf{\Sigma}$ is a Hermitian Toeplitz matrix
- Eigenvalues asymptotically distributed as samples of the Fourier transform of the first row ($J_0(\alpha n)$)
- $F(\omega) = \mathcal{F}\{J_0(\alpha n)\} = 2/\sqrt{\alpha^2 - \omega^2}$



⇒ Can we model $\mathbf{\Sigma}$ as a simpler algebraic matrix?

Spiked covariance matrix

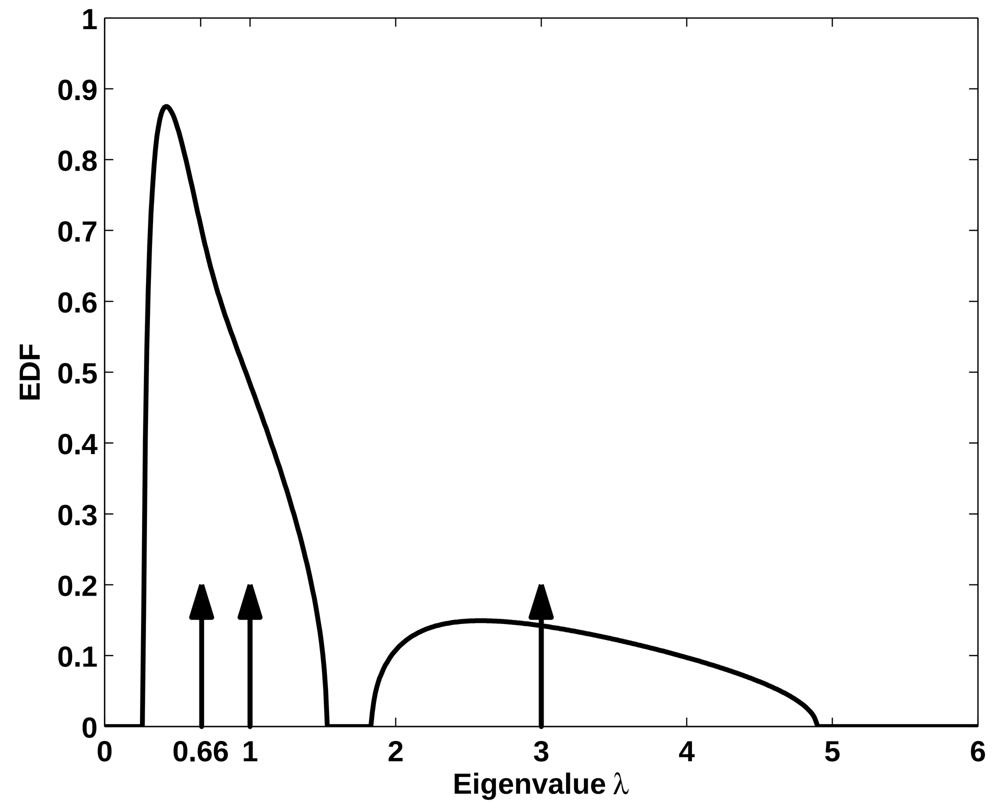
A low rank additive perturbation of a unit power white noise background is known as a spiked covariance matrix [3].

Behavior of sample eigenvalues

- $1 < \gamma_i < (1+\sqrt{c})$: behave as if they were equal to 1
- $\gamma_i > (1+\sqrt{c})$: behave as distinct atoms

Free Multiplicative Convolution

Each impulse in the atomic EDF $f^{\Sigma}(x)$ is replaced by a continuous finite support density function $f^{\mathbf{S_X}}(x)$



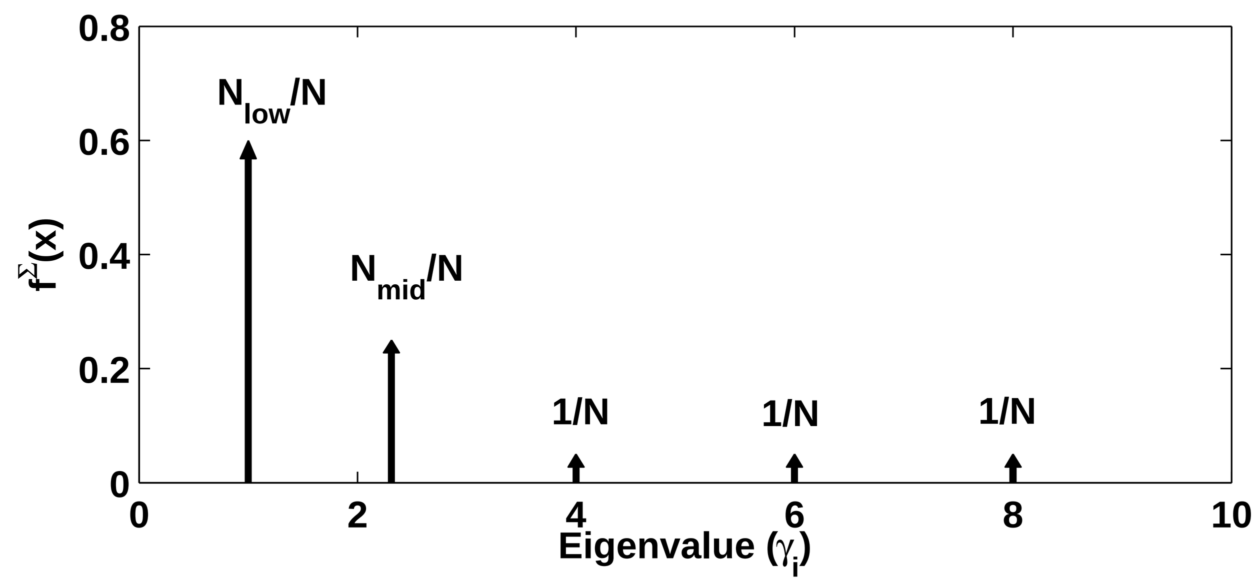
⇒ Can we replace $f^{\Sigma}(x)$ with fewer atoms resulting in much lower order $L_{mz}^{\mathbf{S_X}}$?

Approximating EDF

⇒ Divide eigenvalues into three regions

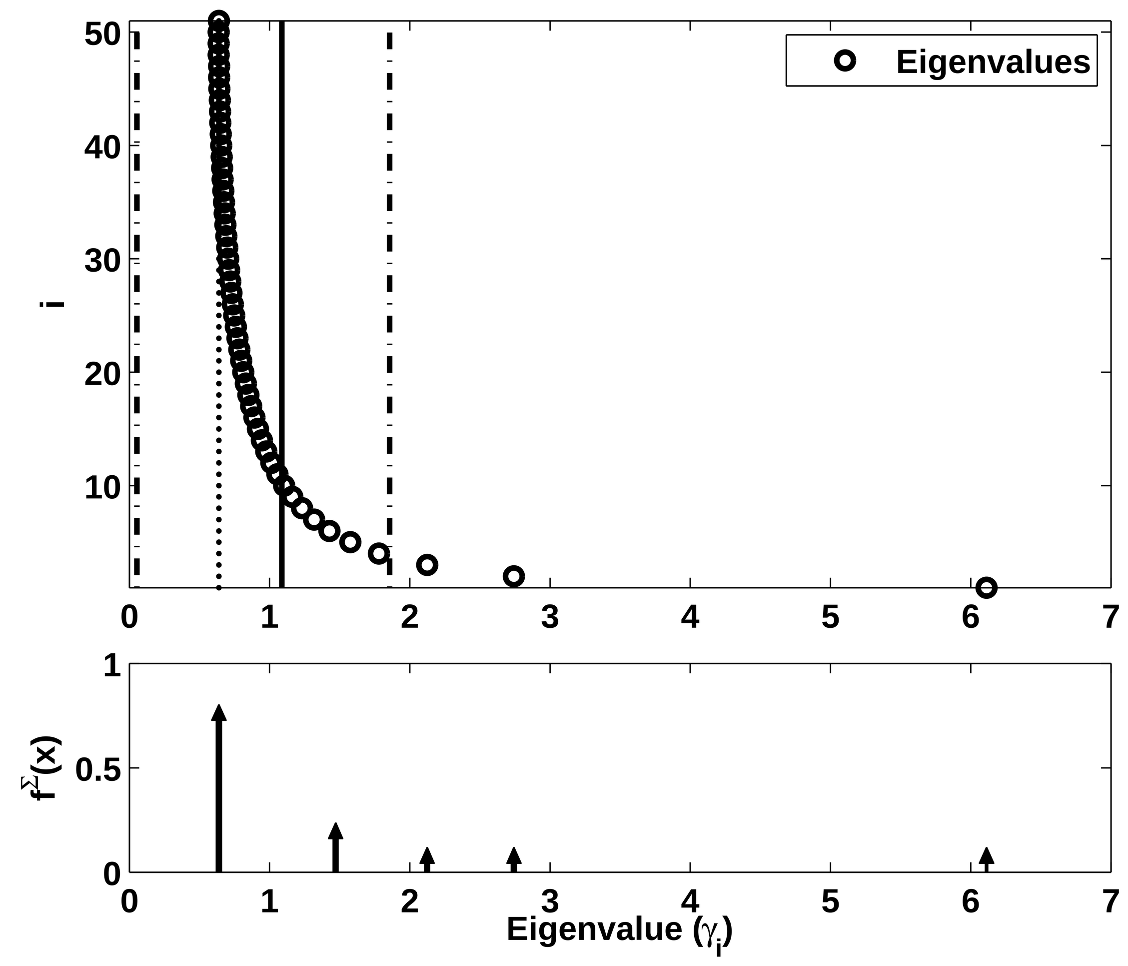
$$1 < \gamma_i < (1+\sqrt{c}) \Rightarrow 1$$
$$(1+\sqrt{c}) < \gamma_i < (1+\sqrt{c})^2 \Rightarrow \gamma_{avg}$$
$$\gamma_i > (1+\sqrt{c})^2 \Rightarrow \gamma_i$$

N_{low} and N_{mid} : # of eigenvalues in the first and mid region.
Results in reduced order atomic density model $\hat{f}^{\Sigma}(x)$



Needs appropriate scaling when white background is not unity power.

Reduced order ensemble EDF

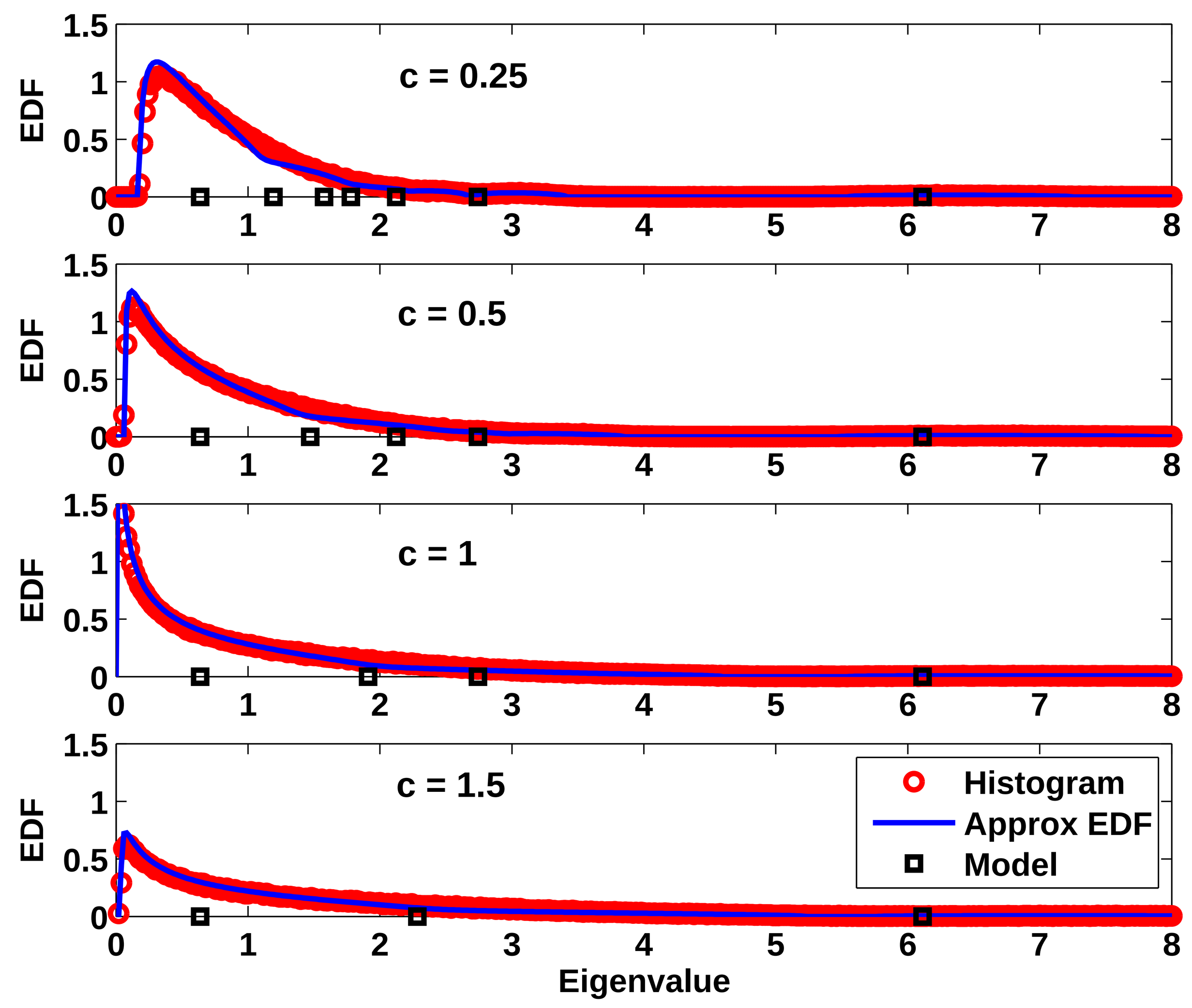


$$\tilde{f}^{\Sigma}(x) = \frac{1}{N}\sum_{\gamma_i \in \Gamma_{dist}} \delta(x-\gamma_i) + \frac{N_{mid}}{N}\delta(x-\gamma_{mid}) + \frac{N_{low}}{N}\delta(x-\gamma_N)$$

Computational savings! N = 50 ⇒ 5 atoms.

Simulation Results

N = 51 element horizontal uniform linear array with ζ=0.5
5000 Monte Carlo simulations



Conclusion

The reduced order model for ensemble eigenvalues accurately approximates the eigenvalue density function of the sample covariance matrix using the polynomial method while realizing a substantial computational savings.

References

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